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COMPUTATIONS OF VECTOR-VALUED SIEGEL MODULAR FORMS

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Abstract. We carry out some computations of vector valued Siegel modular forms of degree two, weight $(k, 2)$ and level one. Our approach is based on Satoh's description of the module of vector-valued Siegel modular forms of weight $(k, 2)$ and an explicit description of the Hecke action on Fourier expansions. We highlight three experimental results: (1) we identify a rational eigenform in a three dimensional space of cusp forms, (2) we observe that non-cuspidal eigenforms of level one are not always rational and (3) we verify a number of cases of conjectures about congruences between classical modular forms and Siegel modular forms.

1. Introduction

Computations of modular forms in general and Siegel modular forms in particular are of great current interest. Recent computations of Siegel modular forms on the paramodular group by Poor and Yuen [20] have led to the careful formulation by Brumer and Kramer [3] of the Paramodular Conjecture, a natural generalization of Taniyama-Shimura. Historically, computations of scalar-valued Siegel modular forms in the 1970s by Kurokawa [16] led to the discovery of the Saito-Kurokawa lift, a construction whose generalizations are still studied. In the 1990s, computations by Skoruppa [28] revealed some striking properties that some Siegel modular forms possess (namely, there are rational eigenforms of weights 24 and 26 in level 1 that span a two-dimensional space of cusp forms). These properties have yet to be explained. This paper is in the same spirit.

We carry out the first systematic computations of spaces of vector-valued Siegel modular forms of degree two and of weight $(k, 2)$. We do this in Sage [25] using a package co-authored by the second author, Raum, Skoruppa and Tornaría [22]. We observe some new phenomena (see Propositions 3.2 and 3.3) and check that the eigenforms we compute satisfy the Ramanujan-Petersson bound (see Proposition 3.1).

We also verify two interesting conjectures on congruences: the first, due to Harder, has been previously verified by Faber and van der Geer [8]; the other,

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We thank Günter Harder for clarifying the case $\delta = 2$ of Conjecture 4.2 for us. We thank Neil Dummigan for many insightful comments regarding Conjecture 4.5. We thank Martin Raum for making available code for computing values of symmetric square L-functions.

due to Bergström, Faber, van der Geer and Harder, has been previously verified by Dummigan [7]. Our approach to verifying these conjectures is to compute Hecke eigenvalues of Siegel modular forms in as direct a manner as possible, using Satoh's concrete description of Siegel modular forms of weight $(k, 2)$. This is a very different approach than the one taken by Faber and van der Geer and we verify cases that they do not (and vice versa). After determining a basis of eigenforms for the space, we use explicit formulas for the Hecke action on Fourier expansions to extract the Hecke eigenvalues. Our main results in this direction are Theorem 4.3 and 4.7, which summarize the cases of the two conjectures that we have verified.

We do not describe the implementation of Siegel modular forms used to carry out these computations but we refer the interested reader to [21] for such a description. The point of this paper is that we were actually able to carry out such computations, have made some new observations based on these computations and have made our data publicly available [24].

2. Vector-valued Siegel modular forms of weight $(k, 2)$

We recall the definition of a Siegel modular form of degree two. We consider the *full Siegel modular group* $\Gamma^{(2)}$ given by

$$\Gamma^{(2)} := \mathrm{Sp}(4,\mathbb{Z}) = \left\{ M \in \mathrm{M}(4,\mathbb{Z})\colon {}^tM \begin{pmatrix} & I_2 \\ -I_2 & \end{pmatrix} M = \begin{pmatrix} & I_2 \\ -I_2 & \end{pmatrix} \right\}.$$

Let

$$\mathbb{H}^{(2)} := \left\{ Z \in \mathrm{M}(2,\mathbb{C}) \ : \ {}^tZ = Z, \mathrm{Im}(Z) > 0 \right\}$$

be the Siegel upper half space of degree 2. For a nonnegative integer j, the space $\mathbb{C}[X,Y]_j$ of homogeneous polynomials of degree j has a GL_2-action given by

$$(A, p) \mapsto A \cdot p := p\big((X,Y)A\big). \tag{2.1}$$

Definition 2.1. Let k, j be nonnegative integers. A *Siegel modular form* of degree 2 and weight (k, j) is a complex analytic function $F\colon \mathbb{H}^{(2)} \to \mathbb{C}[X,Y]_j$ such that

$$F(gZ) := F\big((AZ+B)(CZ+D)^{-1}\big) = \det(CZ+D)^k\,(CZ+D)\cdot F(Z)$$

for all $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma^{(2)}$.

Remark 2.2. The action in (2.1) is a concrete realization of the symmetric power representation Sym^j of GL_2. Definition 2.1 is a concrete description of Siegel modular forms with values in the representation space $\det^k \otimes \mathrm{Sym}^j$ of GL_2. We made these choices in our implementation because it made the multiplication of vector-valued Siegel modular forms easier to implement.

The space of all such functions is denoted $M^{(2)}_{k,j}$, where we suppress j if it is 0. If j is positive F is called *vector-valued*, otherwise it is called *scalar-valued*. We write $M^{(2)}_* := \bigoplus_k M^{(2)}_k$ for the ring of (scalar-valued) Siegel modular forms of degree 2.

Let

$$Q := \left\{ f = [a,b,c] \ : \ a,b,c \in \mathbb{Z},\ b^2 - 4ac \leq 0,\ a \geq 0 \right\}$$

where $[a,b,c]$ corresponds to the quadratic form $aX^2 + bXY + cY^2$.

A Siegel modular form F has a Fourier expansion of the form

$$F(Z) = \sum_{f=[a,b,c]\in Q} C_F(f)\, e\left(a\tau + bz + c\tau'\right).$$

Here $Z := \left(\begin{smallmatrix} \tau & z \\ z & \tau' \end{smallmatrix}\right)$ ($\tau, \tau' \in \mathbb{H}^{(1)}$ and $z \in \mathbb{C}$), $e(x) = e^{2\pi i x}$, the trace of a matrix A is denoted by $\operatorname{tr} A$. The form F is called a *cusp form* if its Fourier expansion is supported on positive-definite elements of Q. The subspace of cusp forms is denoted $S^{(2)}_{k,j}$.

The ring of all vector-valued Siegel modular forms $\bigoplus_{k,j} M^{(2)}_{k,j}$ is not finitely generated. For this reason the symmetric power j is usually fixed. The resulting module is finitely generated over $M^{(2)}_{*}$. We focus exclusively on weight $(k,2)$, where we have a very concrete description of these spaces thanks to work of Satoh.

2.1. **Satoh's Theorem.** The Satoh bracket is a special case of the general Rankin-Cohen bracket construction. Satoh [26] examined the case of weight $(k,2)$. Suppose $F \in M^{(2)}_{k}$ and $G \in M^{(2)}_{k'}$ are two scalar-valued Siegel modular forms. We define the *Satoh bracket* by

$$[F,G]_2 = \frac{1}{2\pi i}\left(\frac{1}{k} G\, \partial_Z F - \frac{1}{k'} F\, \partial_Z G\right) \in M^{(2)}_{k+k',2},$$

where $\partial_Z = \begin{pmatrix} \partial_{Z_{11}} & 1/2\, \partial_{Z_{12}} \\ 1/2\, \partial_{Z_{12}} & \partial_{Z_{22}} \end{pmatrix}$.

In the same paper, Satoh showed that $\bigoplus_k M^{(2)}_{k,2}$ is generated by elements all of which can be expressed in terms of Satoh brackets. More precisely, he showed that

$$\begin{aligned} M^{(2)}_{k,2} =& [E_4,E_6]_2 \cdot M^{(2)}_{k-10} \oplus [E_4,\chi_{10}]_2 \cdot M^{(2)}_{k-14} \oplus \\ & [E_4,\chi_{12}]_2 \cdot M^{(2)}_{k-16} \oplus [E_6,\chi_{10}]_2 \cdot \mathbb{C}[E_6,\chi_{10},\chi_{12}]_{k-16} \oplus \\ & [E_6,\chi_{12}]_2 \cdot \mathbb{C}[E_6,\chi_{10},\chi_{12}]_{k-18} \oplus [\chi_{10},\chi_{12}]_2 \cdot \mathbb{C}[\chi_{10},\chi_{12}]_{k-22}. \end{aligned}$$

Here the forms $E_4, E_6, \chi_{10}, \chi_{12}$ are the generators of the ring of scalar-valued Siegel modular forms described by Igusa [12]. By $\mathbb{C}[A_1,\ldots,A_n]_k$ we mean the module of weight k modular forms that can be expressed in terms of generators $A_1,\ldots,A_n$.

A basis for the space $M^{(2)}_{k,2}$ was computed via a Sage [25] implementation in [22] of an algorithm found in [21].

2.2. **Hecke Operators.** As our interest is in computing Hecke eigenforms, we need to describe how one computes the Hecke action. We give formulas for the image of a Siegel modular form of weight $(k, 2)$ under the operator $T(p^\delta)$. The Hecke operators are multiplicative and so it suffices to understand the image for these operators. The formulas can be found in [11] but we present them here for completeness.

Let F be a Siegel modular form as above and let the image of F under $T(p^\delta)$ have coefficients $C'([a, b, c])$. Then

$$C'([a,b,c]) = \sum_{\alpha+\beta+\gamma=\delta} p^{\beta k+\gamma(2k-1)} \times \sum_{\substack{U \in R(p^\beta) \\ a_U \equiv 0\,(p^{\beta+\gamma}) \\ b_U \equiv c_U \equiv 0\,(p^\gamma)}} (d_{0,\beta} U) \cdot C\left(p^\alpha \left[\frac{a_U}{p^{\beta+\gamma}}, \frac{b_U}{p^\gamma}, \frac{c_U}{p^{\gamma-\beta}}\right]\right) \tag{2.2}$$

where

- $R(p^\beta)$ is a complete set of representatives for $\mathrm{SL}(2,\mathbb{Z})/\Gamma_0^{(1)}(p^\beta)$ where $\Gamma_0^{(1)}(p^\beta)$ is the congruence subgroup of $\mathrm{SL}(2, Z)$ of level p^β;
- for $f = [a, b, c]$, $[a_U, b_U, c_U] = f_U := f\left((X, Y)^t U\right)$;
- $d_{0,\beta} = \begin{pmatrix} 1 & \\ & p^\beta \end{pmatrix}$;
- the $\cdot$ is given by the action defined in (2.1).

We denote the Hecke eigenvalue of a Siegel modular form F under the operator $T(p^\delta)$ by $\lambda_{p^\delta}(F)$. If the space $S^{(2)}_{k,2}$ has dimension d, the Hecke eigenvalues of F are algebraic numbers of degree at most d. The field that contains the Hecke eigenvalues of F is denoted $\mathbb{Q}_F$.

2.3. **Computing Hecke eigenforms.** Fix a space of Siegel modular forms of weight $(k, 2)$ with basis $\{F_1, \dots, F_n\}$, obtained as algebraic combinations of the Igusa generators and Satoh brackets. Because the Hecke operators are a commuting family of linear operators, there is a basis $\{G_1, \dots, G_n\}$ for the space consisting entirely of simultaneous eigenforms.

The forms G_i are determined computationally as follows. First, determine the matrix representation for the Hecke operator $T(2)$ by computing the image under $T(2)$ of each basis element F_i. Build a matrix N that is invertible and whose jth row consists of coefficients of F_j at certain indices $Q_1, \dots, Q_n$. To ensure that N is invertible we pick the indices one at a time, making sure that each choice of index Q_i increases the rank of N. We then construct a matrix M whose jth row consists of coefficients of the image of F_j under $T(2)$ indexed by $Q_1, \dots, Q_n$. Then the matrix representation of $T(2)$ is MN^{-1}.

We compute the Hecke eigenforms using $T(2)$ and we express them in terms of the basis $\{F_1, \dots, F_n\}$. We compute the Hecke eigenvalues λ_{p^δ}

by computing these expressions to high precision and then computing their image under the Hecke operator $T(p^\delta)$ as in (2.2).

2.4. **Hecke eigenvalues, Satake parameters and symmetric polynomials.** Fix a prime p. The Satake isomorphism Ω is a map between the local-at-p Hecke algebra $\mathcal{H}_p$ associated to Γ and a polynomial ring $\mathbb{Q}[x_0, x_1, x_2]^{W_2}$ invariant under the action of the Weyl group. A matrix representation of Ω can be found in [14, 23]. We summarize the relevant results here.

Consider

$$T(p) = \Gamma \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p \end{pmatrix} \Gamma, \qquad T_0(p^2) = \Gamma \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & p^2 & \\ & & & p^2 \end{pmatrix} \Gamma$$

$$T_1(p^2) = \Gamma \begin{pmatrix} 1 & & & \\ & p & & \\ & & p^2 & \\ & & & p \end{pmatrix} \Gamma, \qquad T_2(p^2) = \Gamma \begin{pmatrix} p & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix} \Gamma.$$

The images under Ω of these operators are:

$$\begin{aligned} \Omega(T(p)) &= x_0x_1x_2 + x_0x_1 + x_0x_2 + x_0 \\ \Omega(T_0(p^2)) &= \tfrac{2p-2}{p}\phi_2 + \tfrac{p-1}{p}\phi_1 + \phi_0 \\ \Omega(T_1(p^2)) &= \tfrac{p^2-1}{p^3}\phi_2 + \tfrac{1}{p}\phi_1 \\ \Omega(T_2(p^2) &= \tfrac{1}{p^3}\phi_2 \end{aligned}$$

where

$$\begin{aligned} \phi_0 &= x_0^2x_1^2x_2^2 + x_0^2x_1^2 + x_0^2x_2^2 + x_0^2 \\ \phi_1 &= x_0^2x_1^2x_2 + x_0^2x_1x_2^2 + x_0^2x_1 + x_0^2x_2 \\ \phi_2 &= x_0^2x_1x_2. \end{aligned}$$

Fix a Siegel Hecke eigenform F. It can be shown [1, p. 165] that for any p there exists a triple $(\alpha_{0,p}^F, \alpha_{1,p}^F, \alpha_{2,p}^F) \in (\mathbb{C}^\times)^3/W_2$ with the property that $\Omega(T)|_{x_i \leftarrow \alpha_{i,p}^F} = \lambda_T(F)$, the eigenvalue of F with respect to the Hecke operator T. The numbers α are called the *Satake parameters* of F at p.

We will make use of the following way of expressing $T(p)^2$ in terms of the operators $T_i(p^2)$:

Theorem 2.3 ([8])**.** $T(p)^2 = T_0(p^2) + (p+1)T_1(p^2) + (p^2+1)(p+1)T_2(p^2)$.

3. Computational and Experimental Results

We carry out the computations of particular eigenforms in the following way. Satoh's theorem as described above in Section 2.1 gives a recipe for computing a basis of vector-valued Siegel modular forms of weight $(k, 2)$.

In particular, following [28] we can compute the Fourier expansions of the Igusa generators $E_4, E_6, \chi_{10}, \chi_{12}$. This is done via an explicit map from elliptic modular forms to Siegel modular forms. For each generator we easily computed the part of its Fourier expansion which is supported on positive definite quadratic forms up to discriminant 3000 and on singular quadratic forms $[0, 0, c]$ where $0 \leq c \leq 750$.

Using these four Igusa generators we determine a basis for the space of weight $(k, 2)$ as prescribed by Satoh's theorem: we compute a basis for the modules M_{k-10}, M_{k-14}, M_{k-16} and $\mathbb{C}[E_6, \chi_{10}, \chi_{12}]_{k-16}$, $\mathbb{C}[E_6, \chi_{10}, \chi_{12}]_{k-18}$, $\mathbb{C}[\chi_{10}, \chi_{12}]_{k-22}$. We then form a basis for the space of weight $(k, 2)$ by multiplying each basis above by the Satoh bracket that corresponds to it in Satoh's theorem and end up with eigenforms following the procedure described in Section 2.3. One might pause at the idea of multiplying vector-valued Siegel modular forms but this is precisely the reason why we defined the coefficients of vector-valued Siegel modular forms to be homogeneous polynomials (see Remark 2.2).

For example, we find that a basis for the space of weight $(16, 2)$ is given by

$$\begin{aligned}
G_1 &= E_6[E_4, E_6]_2 - \tfrac{173820100608}{1557539}[E_4, \chi_{12}]_2 + \tfrac{1800409600}{1557539}[E_6, \chi_{10}]_2 \\
G_2 &= [E_4, \chi_{12}]_2 + (\tfrac{5}{8064}\alpha + \tfrac{755}{42})[E_6, \chi_{10}]_2 \\
\overline{G}_2 &= [E_4, \chi_{12}]_2 + (\tfrac{5}{8064}\bar{\alpha} + \tfrac{755}{42})[E_6, \chi_{10}]_2
\end{aligned}$$

where α is a root of $x^2 + 58752x + 858931200$ and $\bar{\alpha}$ is its conjugate. The form G_1 is non-cuspidal (probably Eisenstein) and the other two forms in the basis are cuspidal.

The Hecke eigenvalues that appear in the space $S^{(2)}_{k,2}$ tend to have the largest possible degree, namely the dimension d of the space. For example the forms G_2 and $\overline{G}_2$ above have Hecke eigenvalues in a quadratic field. There are however (very surprisingly!) counterexamples to this; this is analogous to what happens in the scalar-valued spaces $S^{(2)}_{24}$ and $S^{(2)}_{26}$, see [28], but in stark contrast to the situation in degree one (as predicted by Maeda's conjecture).

Consider the weight $(20, 2)$. Let $K = \mathbb{Q}[\alpha]$ be the quadratic number field with minimal polynomial $x^2 - 780288x + 121332695040$, and let $\bar{\alpha}$ denote the conjugate of α in K. The space $S^{(2)}_{20,2}$ is three-dimensional, and a basis

of Hecke eigenforms is given by:

$$H_1 = \chi_{10}[E_4, E_6]_2 - \tfrac{5}{14}E_6[E_4, \chi_{10}]$$
$$H_2 = \chi_{10}[E_4, E_6]_2 + (\tfrac{25}{12241152}\alpha - \tfrac{7685}{15939})E_6[E_4, \chi_{10}]_2 + (\tfrac{-1}{364320}\alpha + \tfrac{674}{759})E_4[E_4, \chi_{12}]_2$$
$$H_3 = \chi_{10}[E_4, E_6]_2 + (\tfrac{25}{12241152}\bar{\alpha} - \tfrac{7685}{15939})E_6[E_4, \chi_{10}]_2 + (\tfrac{-1}{364320}\bar{\alpha} + \tfrac{674}{759})E_4[E_4, \chi_{12}]_2.$$

Note that the first has rational eigenvalues, and the second and third have conjugate quadratic eigenvalues. We checked that each of these forms satisfies the Ramanujan-Petersson conjecture.

Proposition 3.1. *For all k satisfying $14 \le k \le 30$, the Hecke eigenforms in $S^{(2)}_{k,2}$ satisfy the Ramanujan-Petersson conjecture at $p = 2, 3, 5$. More precisely, let $F \in S^{(2)}_{k,2}$ be a Hecke eigenform with eigenvalues λ_p, λ_{p^2} and consider the polynomial*

$$X^4 - \lambda_p X^3 + \left(\lambda_p^2 - \lambda_{p^2} - p^{2k-2}\right) X^2 - p^{2k-1}\lambda_p X + p^{4k-2}.$$

Then all roots $z \in \mathbb{C}$ of this polynomial satisfy $|z| = p^{(2k+j-3)/2}$.

We also looked at the level 1 elliptic modular forms with Hecke eigenvalues in quadratic fields and these fields are different than K. This indicates that H_2 and H_3 are unlikely to be lifts. Therefore the naive generalization of Maeda's conjecture does not hold for $S^{(2)}_{k,2}$. We remark that in all other weights for which we have carried out computations, the naive generalization does indeed hold:

Proposition 3.2. *Let $k \in \{14, 16, 18, 22, 24, 26, 28, 30\}$. Then the characteristic polynomial of the Hecke operator $T(2)$ acting on $S_{k,2}$ is irreducible over $\mathbb{Q}$. If $k = 20$, the characteristic polynomial of the Hecke operator $T(2)$ decomposes over $\mathbb{Q}$ into a linear factor and a quadratic factor.*

We also make note of another interesting computational phenomenon that merits further investigation. It is interesting to us because an analogous phenomenon does not happen in the scalar-valued case. In the scalar-valued case of level 1, there are four kinds of modular forms of even weight: Eisenstein series, Klingen-Eisenstein series, Saito-Kurokawa lifts and cusp forms that are not lifts. The first two are not cuspidal and always have rational coefficients. Compare this fact to the following proposition:

Proposition 3.3. *Let $k \in \{22, 26, 28, 30\}$. The space of modular forms of weight $(k, 2)$ that are not cusp forms is two dimensional but consists of a single Galois orbit.*

The data that make up the proofs of these propositions can be found at [24].

4. Verification of some conjectural congruences

The most famous modular form is arguably

$$\Delta(q) = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - \ldots = \sum_{n=1}^{\infty} \tau(n)q^n.$$

Ramanujan discovered a number of congruences involving the coefficients $\tau(n)$, among which is

$$\tau(p) \equiv p^{11} + 1 \pmod{691} \quad \text{for all primes } p.$$

This is part of a more general phenomenon: if a prime $\ell \geq k-1$ divides the numerator of the zeta-value $\zeta(-k+1)$ (equivalently, the numerator of the Bernoulli number B_k), then the constant term of the Eisenstein series E_k is zero modulo ℓ. This can be interpreted to say that there is a congruence mod ℓ between this Eisenstein series and some cuspidal eigenform of weight k.

As explained in [8] and [9], Deligne's work on attaching families of ℓ-adic Galois representations to Hecke eigenforms of degree one allows us to interpret Ramanujan's congruence as taking place between traces of Frobenius acting on cohomology spaces of local systems.

A more recent development is the construction (initiated by Laumon [18] and Taylor [29] and completed by Weissauer [30]) of families of four-dimensional ℓ-adic Galois representations attached to Siegel modular eigenforms of degree two. It is then natural to ask about generalizations of Ramanujan's congruence to this setting. Building on his study of Eisenstein cohomology for arithmetic groups, Harder stated a conjecture [9] regarding congruences between classical (degree one) eigenforms and Siegel eigenforms of degree two. This statement, which appears below as Conjecture 4.2, was verified in a number of cases by Faber and van der Geer [8], who calculated the number of points on the relevant moduli spaces over finite fields and related them to the Hecke eigenvalues of Siegel modular forms. (This relation was stated as a conjecture in [8], but it has since been proved by van der Geer and Weissauer in most cases. We refer the interested reader to [2] for more details.)

Bergström, Faber and van der Geer have extended this point counting approach to Siegel modular forms of degree three in [2]. At the same time, they formulated another conjectural congruence relating Siegel modular forms of degree two and classical eigenforms, this time via critical values of symmetric square L-functions. We state this below as Conjecture 4.5. It generalizes a result of Katsurada and Mizumoto for scalar-valued Siegel modular forms (see [13]), and a number of vector-valued cases have been proved by Dummigan in [7, Proposition 4.4].

We verify the conjectures by using the data we collected in Section 3 and some custom Sage code that computes critical values of L-functions.

4.1. **Notation.** The conjectures appear in different forms in [9], [8] and [2]. We follow the approach of [2] and adapt it to our notation and the quantities that we compute.

We denote the space of cusp forms of weight r with respect to the group $\Gamma^{(1)} = \mathrm{SL}(2,\mathbb{Z})$ by $S_r^{(1)}$. Suppose $f \in S_r^{(1)}$ is a Hecke eigenform; we denote its Hecke eigenvalue with respect to the operator $T(n)$ by $a_n = a_n(f)$. The spaces $S_r^{(1)}$ are finite dimensional, say of dimension d; according to Maeda's conjecture [10], the eigenvalues a_n are algebraic numbers of degree d. The number field that contains the coefficients is denoted $\mathbb{Q}_f$.

Fix a prime p. For an eigenform $f \in S_r^{(1)}$, let α_0 and α_1 denote the Satake parameters at p and define

$$\mu_{p^\delta}(f) = \alpha_0^\delta + \alpha_0^\delta\alpha_1^\delta \quad \text{for } \delta \geq 1.$$

Similarly, for a Siegel eigenform $F \in S_{k,j}^{(2)}$, let α_0, α_1, α_2 be the Satake parameters at p and define

$$\mu_{p^\delta}(F) = \alpha_0^\delta + \alpha_0^\delta\alpha_1^\delta + \alpha_0^\delta\alpha_2^\delta + \alpha_0^\delta\alpha_1^\delta\alpha_2^\delta \quad \text{for } \delta \geq 1.$$

4.2. ***L*-functions of modular forms.** Let $f(q) = \sum a_n q^n \in S_r^{(1)}$ be an eigenform and consider

$$L(f,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_p \left(1 - a_p p^{-s} + p^{r-1-2s}\right)^{-1}.$$

After introducing the factor at infinity $L_\infty(f,s) = \Gamma(s)/(2\pi)^s$, the completed L-function

$$\Lambda(f,s) = \frac{\Gamma(s)}{(2\pi)^s} L(f,s) = \int_0^\infty f(iy)y^{s-1}dy \tag{4.1}$$

has holomorphic continuation to $\mathbb{C}$ and satisfies the functional equation

$$\Lambda(f,s) = (-1)^{r/2}\Lambda(f, r-s).$$

Its *critical values* occur at $1 \leq t \leq r-1$ (of course, the functional equation implies that it suffices to consider half of this interval).

Manin and Vishik proved that there exist real numbers $\omega_+(f), \omega_-(f)$, called *periods* of f, such that the ratio of the critical values of $\Lambda(f,s)$ and the periods is algebraic. More precisely, define the *algebraic critical values*

$$\widetilde{\Lambda}(f,t) = \begin{cases} \Lambda(f,t)/\omega_+(f) & \text{if } t \text{ is even} \\ \Lambda(f,t)/\omega_-(f) & \text{if } t \text{ is odd.} \end{cases}$$

Theorem 4.1 (Manin-Vishik[19])**.** *If $f \in S_r^{(1)}$ is an eigenform and t is an integer satisfying $1 \leq t \leq r-1$, then $\widetilde{\Lambda}(f,t) \in \mathbb{Q}_f$.*

As explained in [9], the denominators of certain classes in the cohomology groups of local systems on the moduli space of abelian surfaces should be expressed in terms of critical values $\widetilde{\Lambda}(f,t)$. Harder conjectured that the

appearance of certain large primes in these denominators should imply the existence of congruences between eigenvalues of forms of degree one and two.

Conjecture 4.2 (Harder)**.** *Let $f \in S_r^{(1)}$ be a Hecke eigenform with coefficient field $\mathbb{Q}_f$ and let ℓ be an ordinary prime in $\mathbb{Q}_f$ (i.e. such that the ℓ-th Hecke eigenvalue of f is not divisible by ℓ). Suppose $s \in \mathbb{N}$ is such that ℓ^s divides the algebraic critical value $\widetilde{\Lambda}(f,t)$. Then there exists a Hecke eigenform $F \in S_{k,j}^{(2)}$, where $k = r - t + 2$, $j = 2t - r - 2$, such that*

$$\mu_{p^\delta}(F) \equiv \mu_{p^\delta}(f) + p^{\delta(k+j-1)} + p^{\delta(k-2)} \pmod{\ell^s}$$

for all prime powers p^δ.

4.2.1. *Computation of the quantities μ_{p^δ}.* The first step in our numerical verification of the conjecture is to compute, as described in Section 3, the Hecke eigenforms in various weights and their corresponding Hecke eigenvalues $\lambda_p(F)$ and $\lambda_{p^2}(F)$. Once we have those, the second step is to relate the Hecke eigenvalues to the values $\mu_{p^\delta}(F)$. We do this by relating the polynomials that define $\mu_{p^\delta}(F)$ to the expressions of $\lambda_p(F)$ in terms of the Satake parameters $\alpha_0, \alpha_1, \alpha_2$.

Example. Let $\lambda_i(p^2) = \Omega(T_i(p^2))|_{x_i \leftarrow \alpha_i}$. Note $\lambda_2(p^2) = p^{2k+j-6}$ from the definition of the slash operator. Then we have the equations

$$\begin{aligned}
\lambda_p^2 &= \lambda_0(p^2) + (p+1)\lambda_1(p^2) + (p^2+1)(p+1)\lambda_2(p^2) \\
\lambda_{p^2} &= \lambda_0(p^2) + \lambda_1(p^2) + \lambda_2(p^2)
\end{aligned}$$

where the first equation comes from Theorem 2.3.

We compute λ_p, λ_{p^2} and know $\lambda_2(p^2)$. This allows us to solve for $\lambda_0(p^2)$ and $\lambda_1(p^2)$. Then we observe

$$\begin{aligned}
\mu_p &= \lambda_p \\
\mu_{p^2} &= 2\lambda_{p^2} - \lambda_p^2 + 2p^{2k+j-4} \\
\mu_{p^3} &= \left(3\lambda_{p^2} - 2\lambda_p^2 + 3(p+1)p^{2k+j-4}\right)\lambda_p.
\end{aligned}$$

4.2.2. *Computation of the congruence primes ℓ.* We consider the ratios

$$\Lambda(f,2) : \Lambda(f,4) : \cdots : \Lambda(f,r-2) \text{ and } \Lambda(f,1) : \Lambda(f,3) : \cdots : \Lambda(f,r-1)$$

which by Theorem 4.1 are in $\mathbb{Q}_f$.

We compute these ratios of critical values as floating point numbers in Sage [25]. This is done via an implementation of (4.1) due to Dokchitser [5]. We take these floating point numbers and find their minimal polynomial using fplll [4], an implementation of the LLL lattice reduction algorithm wrapped in Sage. We provide an example to illustrate our process and summarize our computations in Table 4.1.

Example. Let $g \in S_{32}^{(1)}$, a two-dimensional space of cusp forms. We will calculate the ratio of critical values

$$\Lambda(g,1) : \Lambda(g,3) : \Lambda(g,5) : \Lambda(g,7) : \Lambda(g,9) : \Lambda(g,11) : \Lambda(g,13) : \Lambda(g,15).$$

As we are interested only in the ratio, we compute

$$\begin{aligned}
&\tfrac{\Lambda(g,1)}{\Lambda(g,1)} : \tfrac{\Lambda(g,3)}{\Lambda(g,1)} : \tfrac{\Lambda(g,5)}{\Lambda(g,1)} : \tfrac{\Lambda(g,7)}{\Lambda(g,1)} : \tfrac{\Lambda(g,9)}{\Lambda(g,1)} : \tfrac{\Lambda(g,11)}{\Lambda(g,1)} : \tfrac{\Lambda(g,13)}{\Lambda(g,1)} : \tfrac{\Lambda(g,15)}{\Lambda(g,1)} = \\
&\qquad 1 : 0.045375\cdots : 0.002369\cdots : 0.000143\cdots : 0.000010\ldots \\
&\qquad\qquad 8.65221\cdots \times 10^{-7} : 8.50052\cdots \times 10^{-8} : 9.23745 \times 10^{-9}
\end{aligned}$$

using a Sage implementation of $\Lambda(g,s)$. Then we find the minimal polynomial of each ratio; e.g., $\frac{\Lambda(3)}{\Lambda(1)}$ has minimal polynomial

$$1254224510x^2 - 471820065x + 18826702.$$

4.2.3. *Checking ordinarity of the primes ℓ.* We use a simple algorithm that is very fast but uses large amounts of storage. Let d be the dimension of $S_r^{(1)}$, and suppose we want to check that ℓ is ordinary for all Hecke eigenforms in $S_r^{(1)}$. We proceed as follows: (a) compute the Victor Miller basis for $S_r^{(1)}$ to a precision of about $d\ell$ coefficients; (b) compute the matrix of the Hecke operator T_ℓ acting on this basis; (c) reduce the matrix modulo ℓ and check whether it is invertible.

This allowed us to verify that most primes ℓ appearing in Table 4.1 are ordinary. The current understanding of the distribution of non-ordinary primes is rather limited, but numerical evidence seems to indicate that they are very rare in level one, so it would be surprising to find a prime $\ell > r$ that divides an algebraic critical value and is non-ordinary.

4.2.4. *Verification of the congruences.* We observe that there are two cases: when $\delta = 1$ and when $\delta > 1$. The difference is that in the first case the congruence reduces to a congruence on the Hecke eigenvalue λ_p while in the second case we require both λ_p and λ_{p^2}. The effect of this difference is that we can verify many more congruences when $\delta = 1$ than when $\delta > 1$; this is due to the number of coefficients needed to compute λ_p as compared to the number needed to compute λ_p and λ_{p^2}.

The way the actual verification works is essentially the same starting from the point where $\mu_{p^\delta}(F)$ and $\mu_{p^\delta}(f)$ have been computed. Each side of the congruence mod ℓ^s is an algebraic number in $\mathbb{Q}_F$ and $\mathbb{Q}_f$ respectively. In the cases we have considered, the exponent s appearing in the conjecture was always 1. We compute the minimal polynomial $m(x)$ of the coefficient a_p of f and the minimal polynomial $M(x)$ of the Hecke eigenvalue λ_p of F. Then we look at the roots of m and M in $\mathbb{F}_\ell$. The conjecture holds if for some choice of root of m and some choice of root of M the congruence holds.

The following statement summarizes our results on Conjecture 4.2.

Theorem 4.3. *Let $r \leq 60$ be a multiple of 4. If $f \in S_r^{(1)}$ is a Hecke eigenform with coefficient field $\mathbb{Q}_f$ and ℓ is an ordinary prime in $\mathbb{Q}_f$ that divides the algebraic critical value $\widetilde{\Lambda}(f, r/2+2)$, then there exists a Hecke eigenform $F \in S_{r/2,2}^{(2)}$ such that*

$$\mu_{p^\delta}(F) \equiv \mu_{p^\delta}(f) + p^{\delta(r/2+1)} + p^{\delta(r/2-2)} \pmod{\ell}$$

for

$$p^\delta \in \{2, 3, 4, 5, 7, 8, 9, 11, 13, 17, 19, 23, 25, 27, 29, 31, 125\}.$$

Proof. For weights $r \leq 28$, we have verified that there are no ordinary primes ℓ dividing $\widetilde{\Lambda}(f, r/2+2)$, so the statement is vacuously true.

For weights $32 \leq r \leq 60$ and

$$p^\delta \in \{2, 3, 4, 5, 7, 9, 11, 13, 17, 19, 23, 25, 29, 31\}$$

we have verified the congruence for all large primes ℓ dividing the algebraic critical value. The results are listed in Table 4.1.

The remaining cases $p^\delta \in \{8, 27, 125\}$ follow from the rest by Proposition 4.8. □

r	t	large $\ell \mid \mathrm{Norm}(\widetilde{\Lambda}(f,t))$	(k, j)	$\dim S_{k,j}^{(2)}$
32	18	211	$(16, 2)$	2
36	20	269741	$(18, 2)$	2
40	22	509 1447	$(20, 2)$	3
44	24	205157	$(22, 2)$	5
48	26	168943	$(24, 2)$	5
52	28	173 929 4261 * 434167	$(26, 2)$	8
56	30	173 1721 38053 1547453	$(28, 2)$	10
60	32	* 325187 * 32210303 * 427092920047	$(30, 2)$	11

TABLE 4.1. A summary of the cases verified numerically for the proof of Theorem 4.3. (The primes ℓ marked with a * have not been checked to be ordinary.)

4.3. **Symmetric square L-functions of modular forms.** It is possible to associate higher-degree L-functions to modular forms, by using various tensorial constructions. We describe the L-function attached to the symmetric square of a modular form.

Fix a Hecke eigenform $f \in S_r^{(1)}$ and let α_p, β_p be the roots of the polynomial $X^2 - a_p X + p^{r-1}$. The associated *symmetric square L-function* is

$$L(\mathrm{Sym}^2 f, s) = \prod_p \left((1-\alpha_p^2 p^{-s})(1-\beta_p^2 p^{-s})(1-\alpha_p\beta_p p^{-s})\right)^{-1}.$$

We take as factor at infinity

$$L_\infty(\mathrm{Sym}^2 f, s) = \frac{\Gamma(s)}{(2\pi)^s}\frac{\Gamma((s+2-r)/2)}{\pi^{(s+2-r)/2}}$$

and set

$$\Lambda(\mathrm{Sym}^2 f, s) = L_\infty(\mathrm{Sym}^2 f, s) L(\mathrm{Sym}^2 f, s).$$

Then $\Lambda(\mathrm{Sym}^2 f, s)$ has holomorphic continuation to $\mathbb{C}$ and satisfies the functional equation

$$\Lambda(\mathrm{Sym}^2 f, s) = \Lambda(\mathrm{Sym}^2 f, 2r-1-s).$$

We define the *algebraic critical values*

$$\widetilde{\Lambda}(\mathrm{Sym}^2 f, t) = \frac{L(\mathrm{Sym}^2 f, t)}{\pi^{2t-r+1}\langle f, f\rangle} \qquad \text{for } t = r, r+2, \ldots, 2r-2, \tag{4.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the Petersson inner product. (It is possible to express the algebraic critical values as quotients of the completed L-function $\Lambda(\mathrm{Sym}^2 f, t)$, which would be closer to the treatment of the usual L-function as given in the previous section. We prefer to take a quotient of $L(\mathrm{Sym}^2 f, t)$ instead, as this is the definition used in much of the existing work on symmetric square L-values.)

Theorem 4.4 (Zagier [31])**.** *If $f \in S_r^{(1)}$ is an eigenform and t is even such that $r \le t \le 2r-2$, then $\widetilde{\Lambda}(\mathrm{Sym}^2 f, t)$ is an algebraic number.*

Moreover, and this is useful for our computations, it can be shown that $\widetilde{\Lambda}(\mathrm{Sym}^2 f, t) \in \mathbb{Q}_f$, see [27].

Conjecture 4.5 (Bergström-Faber-van der Geer-Harder)**.** *Let $f \in S_r^{(1)}$ be a Hecke eigenform with coefficient field $\mathbb{Q}_f$ and let ℓ be a large prime in $\mathbb{Q}_f$. Suppose $s \in \mathbb{N}$ is such that ℓ^s divides the algebraic critical value $\widetilde{\Lambda}(\mathrm{Sym}^2 f, t)$. Then there exists a Hecke eigenform $F \in S_{k,j}^{(2)}$, where $k = t-r+2$, $j = 2r-t-2$, such that*

$$\mu_{p^\delta}(F) \equiv \mu_{p^\delta}(f)(p^{\delta(k-2)}+1) \pmod{\ell^s}$$

for all prime powers p^δ.

The case $j = 0$ concerns scalar-valued Siegel modular forms. The first examples of such congruences were found by Kurokawa [15], who conjectured that they should be governed by certain primes dividing the numerators of algebraic critical values. Kurokawa's conjecture was recently proved by Katsurada and Mizumoto, who even extended these results to the case of scalar-valued Siegel modular forms of arbitrary degree (see [13, Theorem 3.1]).

In the vector-valued setting, the congruence in Conjecture 4.5 was proved for the six rational eigenforms of degree one (weights 12, 16, 18, 20, 22, 26) by Dummigan in [7, Proposition 4.4]. (Dummigan has indicated that it should be possible to extend his Proposition 4.4 to higher weights, using a pullback formula as in Katsurada and Mizumoto [13].)

Remark 4.6. The conjecture does not specify what is meant by a *large* prime ℓ. Dummigan's result uses $\ell > 2r$. In the cases we have verified (see Theorem 4.7 for details), it was sufficient to take $\ell > 2$.

Our numerical verification of Conjecture 4.5 follows the approach of the last section. We highlight only the essential differences.

4.3.1. *Computation of the congruence primes ℓ.* We find the appropriate primes ℓ by computing the algebraic critical values directly from (4.2). The squared-norm $\langle f, f\rangle$ of f can be obtained from the identity

$$\langle f, f\rangle = \frac{(r-1)!}{2^{2r-1}\pi^{r+1}} L(\mathrm{Sym}^2 f, r),$$

so all we require is high-precision evaluation of the symmetric square L-function at various points. For this we use Dokchitser's L-function calculator [5] as wrapped in Sage, as well as some Sage code made available to us by Martin Raum.

Having obtained a sufficiently precise floating point approximation to the algebraic number $\widetilde{\Lambda}(\mathrm{Sym}^2 f, t)$, we then find its minimal polynomial. The congruence primes ℓ are the primes larger than 2 occurring in the factorization of the numerator of the norm of $\widetilde{\Lambda}(\mathrm{Sym}^2 f, t)$.

The critical values we obtain in this way agree with the ones computed by Dummigan in the case of rational eigenforms[1], see Table 1 in [6]. We were also able to verify the case $r = 24$ by comparing our result with the trace of $\widetilde{\Lambda}(\mathrm{Sym}^2 f, 46)$ as obtained (by theoretical means) by Lanphier in [17].

The following statement summarizes our results on Conjecture 4.5.

[1]Dummigan confirmed that a few of the factorizations from Table 1 in [6] are incorrect. Here are the values in question, with their corrected factorizations:

$$\begin{aligned}
&k = 16, r = 3: && 2^{20}/3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13^2 \cdot 17\\
&k = 16, r = 11: && 2^{24} \cdot 839/3^{12} \cdot 5^8 \cdot 7^4 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23\\
&k = 20, r = 11: && 2^{27} \cdot 304477/3^{19} \cdot 5^8 \cdot 7^4 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19 \cdot 23 \cdot 29
\end{aligned}$$

Theorem 4.7. *Let $r \leq 32$. If $f \in S_r^{(1)}$ is a Hecke eigenform with coefficient field $\mathbb{Q}_f$ and $\ell > 2$ is a prime in $\mathbb{Q}_f$ that divides the algebraic critical value $\widetilde{\Lambda}(\mathrm{Sym}^2 f, 2r-4)$, then there exists a Hecke eigenform $F \in S_{r-2,2}^{(2)}$ such that*

$$\mu_{p^\delta}(F) \equiv \mu_{p^\delta}(f)(p^{\delta(r-4)} + 1) \pmod{\ell}$$

for

$$p^\delta \in \{2, 3, 4, 5, 7, 8, 9, 11, 13, 17, 19, 23, 25, 27, 29, 31, 125\}.$$

Proof. For weight $r = 12$, we computed the numerator of the rational number $\widetilde{\Lambda}(\mathrm{Sym}^2\Delta, 20)$ and found it to be -2^{23}, so there are no large primes dividing this algebraic critical value and the statement is vacuously true.

For weights $16 \leq r \leq 32$ and

$$p^\delta \in \{2, 3, 4, 5, 7, 9, 11, 13, 17, 19, 23, 25, 29, 31\}$$

we have verified the congruence for all large primes ℓ dividing the algebraic critical value. The results are listed in Table 4.2.

The remaining cases $p^\delta \in \{8, 27, 125\}$ follow from the rest by Proposition 4.8. □

r	t	odd $\ell \mid \mathrm{Norm}(\widetilde{\Lambda}(\mathrm{Sym}^2 f, t))$	(k, j)	$\dim S_{k,j}^{(2)}$
16	28	373	$(14, 2)$	1
18	32	541 2879	$(16, 2)$	2
20	36	439367	$(18, 2)$	2
22	40	281 286397	$(20, 2)$	3
24	44	2795437 256021114049	$(22, 2)$	5
26	48	4598642018203	$(24, 2)$	5
28	52	4017569791 65593901428085768723	$(26, 2)$	8
30	56	937481 4302719815755987715030485446839	$(28, 2)$	10
32	60	350747 45130901953 432796809552670722149	$(30, 2)$	11

TABLE 4.2. A summary of the cases verified numerically for the proof of Theorem 4.7.

4.4. **Reduction of cubes to primes and squares of primes.** We describe some elementary considerations that allow reducing the case $\delta = 3$ of both conjectures to the cases $\delta = 1$ and $\delta = 2$. For ease of notation in this section, we will write

$$\begin{aligned} g_\delta &= \alpha_0^\delta + \alpha_0^\delta \alpha_1^\delta \\ G_\delta &= \alpha_0^\delta + \alpha_0^\delta \alpha_1^\delta + \alpha_0^\delta \alpha_2^\delta + \alpha_0^\delta \alpha_1^\delta \alpha_2^\delta, \end{aligned}$$

where in the first line α_0 and α_1 are the Satake parameters of $f \in S_r^{(1)}$, while in the second line α_0, α_1 and α_2 are the Satake parameters of $F \in S_{k,j}^{(2)}$.

In the degree one setting, we have the relation $\alpha_0^2 \alpha_1 = p^{r-1}$, which allows us to express g_2 and g_3 in terms of g_1:

$$g_2 = g_1^2 - 2p^{r-1}, \qquad g_3 = g_1(g_1^2 - 3p^{r-1}). \tag{4.3}$$

In the degree two setting, we have the relation $\alpha_0^2 \alpha_1 \alpha_2 = p^{2k+j-3}$, which allows us to express G_3 in terms of G_1 and G_2:

$$G_3 = \frac{1}{2} G_1 \left(-G_1^2 + 3G_2 + 6p^{2k+j-3} \right). \tag{4.4}$$

Proposition 4.8. *In Conjecture 4.2 and Conjecture 4.5, the congruences for the case $\delta = 3$ follow from the congruences for the cases $\delta = 1$ and $\delta = 2$.*

Proof.

(a) Define $h_\delta = p^{\delta(k+j-1)} + p^{\delta(k-2)}$ for all $\delta \geq 1$. Then the congruence in Conjecture 4.2 can be written

$$(C_\delta): \qquad G_\delta \equiv g_\delta + h_\delta \pmod{\ell^s}.$$

It is easily seen that

$$h_2 = h_1^2 - 2p^{2k+j-3}, \qquad h_3 = h_1(h_1^2 - 3p^{2k+j-3}). \tag{4.5}$$

(Observe the similarities between these equations and (4.3).)

We assume that the congruences (C_1) and (C_2) hold. Using Equations (4.4), (4.3) and (4.5) (in this order), we compute

$$\begin{aligned} G_3 &\equiv \frac{1}{2}(g_1 + h_1) \left(-(g_1 + h_1)^2 + 3(g_2 + h_2) + 6p^{2k+j-3} \right) \\ &= \frac{1}{2}(g_1 + h_1) \left((-g_1^2 + 3g_2) - 2g_1 h_1 + (-h_1^2 + 3h_2 + 6p^{2k+j-3}) \right) \\ &= (g_1 + h_1) \left(g_1^2 - 3p^{r-1} - g_1 h_1 + h_1^2 \right) \\ &= g_1^3 - 3p^{r-1} g_1 + h_1^3 - 3p^{r-1} h_1 \\ &= g_3 + h_3, \end{aligned}$$

after noting that, under the conditions of Conjecture 4.2, the weight parameters r, k and j are related by $r - 1 = 2k + j - 3$.

(b) The calculation is similar to the previous part. We let $h_\delta = p^{\delta(k-2)} + 1$ for all $\delta \geq 1$. The congruence in Conjecture 4.5 takes the form

$$(C'_\delta): \qquad G_\delta \equiv g_\delta h_\delta \pmod{\ell^2}.$$

We easily see that

$$h_2 = h_1^2 - 2p^{k-2}, \qquad h_3 = h_1(h_1^2 - 3p^{k-2}).$$

Assuming that congruences (C_1') and (C_2') hold, we obtain

$$\begin{aligned}G_3 &\equiv \frac{1}{2} g_1 h_1 \left(-g_1^2 h_1^2 + 3(g_1^2 - 2p^{r-1})(h_1^2 - 2p^{k-2}) + 6p^{2k+j-3}\right)\\ &= g_1 h_1 \left(g_1^2 h_1^2 - 3p^{k-2} g_1^2 - 3p^{r-1} h_1^2 + 9p^{2k+j-3}\right)\\ &= g_1 h_1 \left(g_1^2 - 3p^{r-1}\right)\left(h_1^2 - 3p^{k-2}\right)\\ &= g_3 h_3,\end{aligned}$$

where we used the relation $r = k + j$, valid under the conditions of Conjecture 4.5.

□

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